Commodities

Mini 4 – 1999
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Assignment #2 – Due Thursday April 15

PART (A) – Equilibrium Risk/Return Models and the Derivatives Models
This question looks at a parametric example of Jaganathan (1985) or the multi-period versions of the Grauer and Litzenberger. Unlike these papers we will specify an endowment and preferences to pin down commodity prices as well as the risk premium issue. From this we can think about how to construct equilibrium versions of the various Schwartz models. The last part of the question introduces the production / inventory topics that dominate the last part of the class. Parts of the question are difficult. Chat with your colleagues or see me for hints if needed. If you find any typos, let me know and I can notify others promptly.

The economy is an infinite horizon and discrete time. There is one (representative consumer).

° There are K=2 non-storable goods.
° Consumption is \( c_t = (c_{t0}, c_{t1}) \)
° All prices are expressed in terms of good 0 (or the price of good zero is normalized to 1).
° Endowments of \( e_t = (e_{t0}, e_{t1}) \). You can specify the endowment process in various ways to specialize the model. Here are three DIFFERENT processes that I will mention later. (Feel free to modify these as needed).
° PROCESS (1) – i.i.d.
\[
\ln e_{t0} = \ln e_{t-1,0} + \mu_0 + \sigma_0 e_0 \\
\ln e_{t1} = \ln e_{t-1,1} + \mu_1 + \sigma_1 e_1
\]
° PROCESS (2) – AR1
\[
\ln e_{t-1,0} = \bar{e}_0 + \kappa \ln e_{t-1,0} + \sigma_0 e_0 \\
\ln e_{t-1,1} = \bar{e}_1 + \kappa \ln e_{t-1,1} + \sigma_1 e_1
\]
° PROCESS (3) – AR1
\[
\ln e_{t-1,0} = \ln e_{t-1,0} + \mu_0 + \sigma_0 e_0 \\
\ln e_{t-1,1} = (1 - \kappa)\bar{e}_1 + \kappa \ln e_{t-1,1} + \sigma_1 e_1
\]
where $\epsilon = \rho \epsilon_0 + \xi_1$, and $\epsilon_0, \epsilon_1, \xi_1$ are i.i.d. N(0,1). $\epsilon_0, \xi_1$ are uncorrelated. The correlation between $\epsilon_0$ and $\epsilon_1$ is $\rho$.

Preferences are additively separable (the consumption and portfolio problems are separable.)

$$E_0 \sum_{t=0}^{\infty} \beta^t U(c_t, \phi_t, P_t)$$

V(W_t, P_t) = \max_{c_t} \left[ v(c_t) \right] = c_{t0} + P_t c_{t1} \leq W_t$

The specific preferences we will consider are log normal and CRRA. If $\gamma = 1$, the CRRA preferences are $\ln(V)$.

$$U^\gamma = (1 - \gamma)^{-1} V^{1-\gamma}$$

The preferences over goods are Cobb-Douglas. Note the preference are homothetic.

$$v(c_t) = (c_{t0})^{1-\alpha} (c_{t1})^{\alpha}$$

Normalize the price of good zero to be one. The commodity price of interest is the price of good one ($P_t$).

Assume asset markets are complete, so the unique pricing kernel comes from the Euler equation of the representative agent.

Questions

1. Given endowment and price, solve for the optimal commodity demands.
2. Given that the representative agent must consume her endowment, solve for the equilibrium commodity price given the endowment.
3. What sort of restriction is needed on the endowment process given that this is an infinitely lived agent.
4. What is the stochastic process for price under endowment process (1), (2), and (3). Can you relate these to the assumptions in the standard assumptions made in the derivatives papers. (I am not sure here but I think (2) $\rightarrow$ Schwartz Model 1 and (3) $\rightarrow$ Smith and Schwartz. Are there other endowment assumptions that give you the other Schwartz models we discussed?)
5. Use the homotheticity to show that $U(W_t, P_t) = U(W_t^{1-\gamma})$. To do this, plug the optimal demands into $v$. Express $I_t$ in terms of the ratio of the endowments of the two goods. Provide an interpretation.
6. Specialize to the case of log $\gamma = 1$. What is the pricing kernel.
7. What is the condition (on the endowment process) required for the risk-free rate to be constant (and the term-structure of interest rates to be flat).
8. In this part you can use the forward contract or the futures contract on $e_t$. I am not sure which one is easier to work with.
9. Decompose the forward price into expected spot price plus a risk premium.
10. Since everything is linear and normal in the logs, you can solve explicitly for the forward price, expected spot price and premium.
11. Try this for each of the suggested endowment processes.
12. What are the conditions needed to have a constant risk premium. Why is a time-variation in the risk premium a problem for the Schwartz approach.
9. Recall the Fama-French regression of the “basis” into the “expected change” and “risk premium.” I think it is easier to work in the log of the prices.

\[ p_T - p_t = a_1 + b_1 (f_{i,T} - p_t) + \eta_1 \]
\[ f_{i,T} - p_t = a_2 + b_2 (f_{i,T} - p_t) + \eta_2 \]

Under process (2), calculate the regression coefficients. What can we learn about the economy from the Fama-French regressions.

10. Briefly (or in as much detail as you like), consider how your results will change as we vary the risk aversion parameter \( \gamma \).

11. Think about how the model will change if the one (or both) of the commodities are storable. How do things change if one (or both) of the commodities can be converted into the other. How is this model similar / different than models of international asset pricing. (Note: “think about” does not require you to turn in any answers for this part).

**PART (B) – Other questions**

1. We did not get a chance to discuss the results Jaganathan (1985). Briefly compare and contrast the results he obtains to the standard “puzzles” in asset pricing.

2. Use Ito’s lemma to show that if \( dS / S = \mu dt + \sigma dz \), then

\[ \ln(S_T) \sim \mathcal{N}(\ln(S_0) + (\mu - 0.5\sigma^2)T, \sigma^2 T) \]

3. Show that if \( x \sim \mathcal{N}(\mu, \sigma^2) \), then \( E[\exp(x)] = \exp(\mu + 0.5\sigma^2) \). The trick here is to write down the expectation integral explicitly. You can then use the “complete the square” trick to get an integral that integrates to one. (If you are bold, derive the result for a vector of \( m \) jointly normal random variables).

4. Consider Schwartz model 2 (see his paper for the notation). Construct a hedge portfolio from the spot and two derivative contracts. Does this eliminate the need to make a “price of risk” assumption. Does this help?

5. [I am assuming you have already done this…] Using a spreadsheet (or any other numerical package like Matlab etc.), implement the Schwartz model 1 and 2 for crude oil. In addition, implement the Brennan model (Schwartz 1.5) with the functional form \( y(t, S) = a + bS \). In this case, you will need to solve for the Forward prices using M.C.S. In each case: What types of forward curves can each of the models produce (or not produce)?